

On plane waves in diluted relativistic cold plasmas

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Abstract

We briefly report on some exact results [1] regarding plane waves in a relativistic cold plasma. If the plasma, initially at rest, is reached by a transverse plane electromagnetic travelling-wave, then its motion has a very simple dependence on this wave in the limit of zero density, otherwise can be determined by an iterative procedure whose accuracy decreases with time or the plasma density. Thus one can describe in particular the impact of a very intense and short laser pulse onto a plasma and determine conditions for the *slingshot effect* [2] to occur. The motion in vacuum of a charged test particle subject to a wave of the same kind is also determined, for any initial velocity.

1 Introduction

The amazing developments of laser technologies today allow the production of very intense (hundreds of TeraWatts), coherent electromagnetic (EM) waves concentrated in very short pulses (tens of femtoseconds). The interaction of such laser pulses with isolated electric charges or with continuous matter is characterized by so fast, huge and highly nonlinear effects that traditional approximation schemes are seriously challenged. Even if the initial state of matter is not of plasma type, the huge kinetic energy κ transferred to the electrons almost immediately ionizes matter locally into a plasma¹ (thereafter quantum effects are completely negligible). The kinetic energies transferred to electrons and ions are also many

¹Each level of ionization (first, second,...) is practically complete if the associated Keldysh parameter $\Gamma_i := \sqrt{U_i/\kappa}$ (U_i is the associated ionization potential) fulfills $\Gamma_i \ll 1$ [3, 4].

orders of magnitude above the typical values of the thermal spectrum (even if the temperature is millions of °K!), therefore classical relativistic Magneto-Fluid-Dynamics (MFD) at zero temperature, with its full nonlinearity, is a perfectly accurate framework while the pulse is passing, and also afterwards as long as dissipation has not produced significant effects.

Due to the extremely high number of electrons in a plasma, moderate displacements w.r.t. ions generate huge electric fields that may in turn lead to extreme acceleration of charged particles. Understanding the underlying collective effect mechanisms would be crucial for many scopes, from shedding light on some violent astrophysical phenomena to conceiving a completely new kind of particle accelerators. Today accelerators are used in particular for:

1. nuclear medicine, cancer therapy (PET, electron/proton therapy,...);
2. research in structural biology;
3. research in materials science;
4. food sterilization;
5. research in nuclear fusion (inertial fusion);
6. transmutation of nuclear wastes;
7. research in high-energy particle physics.

Past and present-day acceleration technology (cyclotrons, synchrotrons, etc) relies on the interaction of radio-frequency (RF) EM waves with ‘few’ charged particles (those one wishes to accelerate) over long distances. It has been developed for purpose 7, but has more recently found very important applications also for the other ones. By now it is close to its structural limits. The present or recent most powerful accelerators (LHC and its predecessor LEP at CERN, SLAC in Stanford, Tevatron at Fermilab), which accelerate(d) particles up to energies of 100 - 1000 GeV, are (or were) already very big and expensive. In 2011 Tevatron closed because of budget cuts; building higher energy accelerators would be prohibitive. Much lower energies (100 - 300 MeV) are needed for other uses; but the still too large machines and high costs prevent the use on a large scale. For instance, the CNAO center in Pavia - one of the few centers for cancer treatment by hadron therapy - is based on a 25-meters diameter synchrotron which has costed about 100 million Euro.

A lot of theoretical efforts are being made to conceive and construct new types of ‘table-top’ plasma-based acceleration machines, at least for energies of the order $100 \div 1000$ MeV. A beam of electrons of about 200 MeV with very little energetic and angular spread has

been produced within a distance of few mm through the so-called Laser Wake Field (LWF) mechanism [5] in the bubble-regime [6] in experiments at the *École Polytechnique* [7]. The involved accelerations are thousands of times the ones generated by RF-based accelerators. Present theoretical research on the subject is dominated by numerical resolution programs of the MFD equations (‘particle-in-cell’ simulations, etc.) or their substitution by (sometimes over-) simplified models. This leads to a qualitative understanding of some phenomena, at best. Substantial progress in a rigorous analytical study of the MFD equations would be highly welcome.

Here we briefly report about recent exact results [1] applying to the differential (sect. 2) and equivalent integral equations (sect. 5) ruling a relativistic cold plasma after the plane-wave Ansatz. If the plasma, initially at rest, is reached by a transverse plane EM travelling-wave, then the solution has a very simple dependence on the EM potential in the limit of zero density (sect. 3); otherwise the zero-density solution is a good approximation of the real one as long as the back-reaction of the charges on the EM field can be neglected (i.e. for a time lapse decreasing with the plasma density), and can be corrected into better and better ones by an iterative procedure. In sect. 6 we sketch how to use these results to describe the impact of an ultra-intense and ultrashort laser pulse with a plasma and determine conditions under which a new phenomenon named *slingshot effect* [2] should occur. The general motion of a charged test particle in the above EM wave is determined in sect. 4.

We first fix the notation and recall the basic equations. We denote as $x = (x^\mu) = (x^0, \mathbf{x}) = (ct, \mathbf{x})$ the spacetime coordinates (c is the light velocity), $(\partial_\mu) \equiv (\partial/\partial x^\mu) = (\partial_0, \nabla)$, as $(A^\mu) = (A^0, \mathbf{A})$ the EM potential, as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ the EM field, and consider a collisionless plasma composed by $k \geq 2$ types of charged particles (electrons, ions). For $h=1, \dots, k$ let m_h, q_h be the rest mass and charge of the h -th type of particle (as usual, $-e$ the charge of electrons), $\mathbf{v}_h(x)$, $n_h(x)$ respectively the 3-velocity and the density (number of particles per unit volume) of the corresponding fluid element located in position \mathbf{x} at time t . It is convenient to use dimensionless variables like

$$\boldsymbol{\beta}_h := \mathbf{v}_h/c, \quad \gamma_h := 1/\sqrt{1-\boldsymbol{\beta}_h^2}$$

$$\text{4-vector velocity: } u_h = (u_h^\mu) = (u_h^0, \mathbf{u}_h) := (\gamma_h, \gamma_h \boldsymbol{\beta}_h) = \left(\frac{p_h^0}{m_h c}, \frac{\mathbf{p}_h}{m_h c} \right)$$

(then $u_h^\mu u_{h\mu} = 1$, $\gamma_h = u_h^0 = \sqrt{1+\mathbf{u}_h^2}$, $\boldsymbol{\beta}_h = \mathbf{u}_h/\gamma_h$), and

$$\text{4-vector current density: } (j^\mu) = (j^0, \mathbf{j}) = \left(\sum_{h=1}^k q_h n_h, \sum_{h=1}^k q_h n_h \boldsymbol{\beta}_h \right)$$

The Eulerian and Lagrangian descriptions of an observable are related by

$$\tilde{f}_h(x^0, \mathbf{X}) = f_h[x^0, \mathbf{x}_h(x^0, \mathbf{X})] \quad \Leftrightarrow \quad f_h(x^0, \mathbf{x}) = \tilde{f}_h[x^0, \mathbf{X}_h(x^0, \mathbf{x})], \quad (1)$$

where $\mathbf{x}_h(x^0, \mathbf{X})$ is the position at time t of the h -fluid element initially located in \mathbf{X} , one requires $\mathbf{x}_h \in C^1(\mathbb{R}^4)$ and the inverse $\mathbf{X}_h(x^0, \cdot) : \mathbf{x} \mapsto \mathbf{X}$ of $\mathbf{x}_h(x^0, \cdot) : \mathbf{X} \mapsto \mathbf{x}$ to exist. Conservation of the particles of the h -th fluid reads

$$\tilde{n}_h \left| \frac{\partial \mathbf{x}_h}{\partial \mathbf{X}} \right| = \widetilde{n_{h0}}(\mathbf{X}) \quad \Leftrightarrow \quad n_h \left| \frac{\partial \mathbf{X}_h}{\partial \mathbf{x}} \right|^{-1} = n_{h0} \quad (2)$$

and implies the continuity equation

$$\frac{dn_h}{dx^0} + n_h \nabla \cdot \boldsymbol{\beta}_h = \partial_0 n_h + \nabla \cdot (n_h \boldsymbol{\beta}_h) = 0; \quad (3)$$

here $\frac{d}{dx^0} := \frac{d}{cdt} = \partial_0 + \beta_h^l \partial_l = \frac{v_h^\mu}{\gamma_h} \partial_\mu$ is the *material* derivative for the h -th fluid, rescaled by c . In the CGS system Maxwell's equations and the (Lorentz) equations of motion of the fluids in Lorentz-covariant formulation read

$$\square A^\nu - \partial^\nu (\partial_\mu A^\mu) = \partial_\mu F^{\mu\nu} = 4\pi j^\nu, \quad (4)$$

$$-q_h u_{h\mu} F^{\mu\nu} = m_h c^2 u_{h\mu} \partial^\mu u_h^\nu \quad (5)$$

(Eulerian description); (5) $_{\nu=0}$ follows also from contracting (5) $_{\nu=l}$ with u_h^l , $l=1, 2, 3$. Dividing (5) $_{\nu=l}$ by γ_h gives the familiar 3-vector formulation of (5)

$$q_h \left(\mathbf{E} + \frac{\mathbf{v}_h}{c} \wedge \mathbf{B} \right) = \partial_t \mathbf{p}_h + \mathbf{v}_h \cdot \nabla \mathbf{p}_h = \frac{d\mathbf{p}_h}{dt} \quad (6)$$

in terms of the electric and magnetic fields $E^l = F^{l0} = -\partial_0 A^l - \partial_l A^0$, $B^l = -\frac{1}{2} \varepsilon^{lkn} F^{kn} = \varepsilon^{lkn} \partial_k A^n$. Given the initial momenta $\widetilde{\mathbf{p}_{h0}}$ and densities $\widetilde{n_{h0}}$ in (2) the unknowns are $A^\mu, \mathbf{x}_h, \mathbf{u}_h$ and the equations to be solved are (4-5) and

$$\partial_0 \mathbf{x}_h(x^0, \mathbf{X}) = \boldsymbol{\beta}_h[x^0, \mathbf{x}_h(x^0, \mathbf{X})]. \quad (7)$$

2 Lorentz-Maxwell equations for plane waves

We restrict our attention to solutions such that for all h :

$$A^\mu, n_h, \mathbf{u}_h \quad \text{depend only on } z \equiv x^3, x^0 \quad (\text{plane wave Ansatz}), \quad (8)$$

$$\begin{aligned} A^\mu(x^0, z) &= 0, & \mathbf{u}_h(x^0, z) &= \mathbf{0}, & \text{if } x^0 \leq z, \\ \exists \widetilde{n_{h0}}(z) \quad \text{such that } \sum_{h=1}^k q_h \widetilde{n_{h0}} &\equiv 0, & n_h(x^0, z) &= \widetilde{n_{h0}}(z) & \text{if } x^0 \leq z. \end{aligned} \quad (9)$$

Eq. (8-9) entail a partial gauge-fixing, imply $\mathbf{B} = \mathbf{B}^\perp = \hat{\mathbf{z}} \wedge \partial_z \mathbf{A}^\perp$, $\mathbf{E}^\perp = -\partial_0 \mathbf{A}^\perp$,

$$\mathbf{E}(x) = \mathbf{B}(x) = \mathbf{0}, \quad \mathbf{x}_h(x) = \mathbf{x} \quad \text{if } x^0 \leq z, \quad (10)$$

$-\mathbf{A}^\perp(x^0, z) = \int_z^{x^0} d\eta \mathbf{E}^\perp(\eta, z) = \int_{-\infty}^{x^0} d\eta \mathbf{E}^\perp(\eta, z)$, so that \mathbf{A}^\perp becomes a *physical observable*, and the existence of the limits $n_h(-\infty, Z) = \widetilde{n}_{h0}(Z)$, $\mathbf{x}_h(-\infty, \mathbf{X}) = \mathbf{X}$. Hence we can adopt $-\infty$ as the ‘initial’ time in the Lagrangian description. The map $\mathbf{x}_h(x^0, \cdot) : \mathbf{X} \mapsto \mathbf{x}$ is invertible iff $z_h(x^0, Z)$ is strictly increasing w.r.t. $Z \equiv X^3$ for each fixed x^0 . We shall abbreviate $Z_h \equiv X_h^3$. Eq. (2) becomes

$$\tilde{n}_h(x^0, Z) \partial_Z z_h(x^0, Z) = \widetilde{n}_{h0}(Z), \quad \Leftrightarrow \quad n_h = n_{h0} \partial_z Z_h. \quad (11)$$

$\partial_0 Z = 0$ in the Eulerian description gives $\frac{dZ_h}{dx^0} = \partial_0 Z_h + \beta_h^z \partial_z Z_h = 0$ and by (11)

$$n_{h0} \partial_0 Z_h + n_h \beta_h^z = 0. \quad (12)$$

As known, eq. (5) $_{\nu=x,y}$ amounts to $\frac{d}{dx^0}(m_h c^2 \mathbf{u}_h^\perp + q_h \mathbf{A}^\perp) = 0$, which implies $m_h c^2 \tilde{\mathbf{u}}_h^\perp + q_h \tilde{\mathbf{A}}^\perp = C(\mathbf{X})$; by (9) $C(\mathbf{X}) \equiv 0$, whence

$$\tilde{\mathbf{u}}_h^\perp = \frac{-q_h}{m_h c^2} \tilde{\mathbf{A}}^\perp \quad \Leftrightarrow \quad \mathbf{u}_h^\perp = \frac{-q_h}{m_h c^2} \mathbf{A}^\perp, \quad (13)$$

which explicitly gives \mathbf{u}_h^\perp in terms of \mathbf{A}^\perp . Eq. (4) and the remaining (5) become

$$(4)_{\nu=0} : \quad \partial_z E^z = 4\pi \sum_{h=1}^k q_h n_h, \quad (14)$$

$$(4)_{\nu=z} : \quad \partial_0 E^z = -4\pi \sum_{h=1}^k q_h n_h \beta_h^z, \quad (15)$$

$$(4)_{\nu=x,y} : \quad [\partial_0^2 - \partial_z^2] \mathbf{A}^\perp = 4\pi \underbrace{\sum_{h=1}^k q_h n_h \beta_h^\perp}_{-\frac{4\pi}{c^2} \mathbf{A}^\perp \sum_{h=1}^k \frac{q_h^2 n_h}{m_h \gamma_h}}, \quad (16)$$

$$(5)_{\nu=0} : \quad \frac{d\gamma_h}{dx^0} - \frac{q_h u_h^z E^z}{\gamma_h m_h c^2} - \frac{q_h^2 \partial_0 (\mathbf{A}^\perp)^2}{2\gamma_h m_h^2 c^4} = 0 \quad (17)$$

$$(5)_{\nu=z} : \quad \frac{du_h^z}{dx^0} - \frac{q_h E^z}{m_h c^2} + \frac{q_h^2 \partial_z (\mathbf{A}^\perp)^2}{2\gamma_h m_h^2 c^4} = 0 \quad (18)$$

The independent unknowns in (14-18) are $\mathbf{A}^\perp, u_h^z, E^z$ (all observables).

2.1 Magnetic and ponderomotive force

The term $F_{hm}^z := -\partial_z q_h^2 \mathbf{A}^{\perp 2} / 2\gamma_h m_h c^2$ in (18) is the longitudinal magnetic part $q_h(\boldsymbol{\beta}_h \wedge \mathbf{B})^z$ of the Lorentz force [cf. (6)]; $U_{hm} := -q_h^2 \mathbf{A}^{\perp 2} / 2m_h c^2$ acts like a ‘time-dependent potential energy’. For fixed x^0 let $\bar{z} < x^0$ the right extreme of $\text{supp} \mathbf{A}^\perp$: then $\mathbf{A}^{\perp 2}(x^0, z) = 0$ for $z \geq \bar{z}$, whereas $\mathbf{A}^{\perp 2}(x^0, z)$ is positive and necessarily strictly decreasing for z in a suitable interval $[z', \bar{z}]$. Then F_{hm}^z acts as a positive longitudinal force on *all* the electric charges located at $z \in [z', \bar{z}]$.

In the prototypical cases of a modulated monochromatic transverse wave

$$\begin{aligned} \mathbf{E}^\perp(x^0, z) &= \boldsymbol{\epsilon}^\perp(x^0 - z), & \boldsymbol{\epsilon}^\perp(\xi) &= \epsilon_s(\xi) \boldsymbol{\epsilon}_o^\perp(\xi), \\ \boldsymbol{\epsilon}_o^\perp(\xi) &= \hat{\mathbf{x}} \cos k\xi, & \boldsymbol{\epsilon}_p^\perp(\xi) &= \hat{\mathbf{x}} \sin k\xi & (\text{linearly polarized}), \text{ or} \\ \boldsymbol{\epsilon}_o^\perp(\xi) &= \hat{\mathbf{x}} \cos k\xi + \hat{\mathbf{y}} \sin k\xi, & \boldsymbol{\epsilon}_p^\perp(\xi) &= -\frac{1}{k} \boldsymbol{\epsilon}_o^{\perp'} & (\text{circularly polarized}), \end{aligned} \quad (19)$$

with amplitude not varying significantly over $\lambda := 2\pi/k$, e.g. $\lambda |\epsilon_s'(\xi)/\epsilon_s(\xi)| \leq \delta \ll 1$ for all ξ , then $\mathbf{A}^\perp(x^0, z) = \left\{ \frac{1}{k} \epsilon_s \left[\boldsymbol{\epsilon}_p^\perp + O(\delta) \right] \right\} (x^0 - z)$. The *ponderomotive force* $F_{hp}^z := \langle F_{hm}^z \rangle$ ($\langle \cdot \rangle$ stands for the average over a period λ) plays a crucial role in the LWF acceleration and in the slingshot effect. Up to $O(\delta)$ one finds

$$\begin{aligned} F_{hm}^z &= [\mu_h (\epsilon_s \boldsymbol{\epsilon}_p^\perp)^{2'}] (x^0 - z), & F_{hp}^z &= \frac{1}{2} [\mu_h (\epsilon_s^2)'] (x^0 - z) & \text{lin. polarized,} \\ F_{hm}^z &= [\mu_h (\epsilon_s \boldsymbol{\epsilon}_p^\perp)^{2'}] (x^0 - z) = [\mu_h (\epsilon_s^2)'] (x^0 - z) \equiv F_{hp}^z & & \text{circ. polarized;} \end{aligned} \quad (20)$$

here we abbreviated $\mu_h := \lambda^2 q_h^2 / 8\pi^2 \gamma_h m_h c^2$, and we have used that $\boldsymbol{\epsilon}_p^{\perp 2} = 1$ for circular polarization. Hence the ponderomotive force $F_{hp}^z(x^0, z)$ is positive (resp. negative) for *all* h if $\epsilon_s^2(\xi)$ is increasing (resp. decreasing) at $\xi := x^0 - z$ (fig. 1 - right A). Therefore, while the transversal motion is oscillatory with period λ and averages to zero, the longitudinal motion is ruled in average by ϵ_s on the much larger scale l ; in the case of a linearly polarized wave the rapid spatial oscillations of $\boldsymbol{\epsilon}_p^{\perp 2}$ have the additional effect of modulating the densities, especially of the electrons (the lightest particles), into equi-spatiated bunches (see fig. 1 - right B,C). Moreover, for large amplitudes ($u_h \gg 1$) the direction of \mathbf{v}_h is close to the longitudinal one for most of the time.

3 The zero-density solutions

Proposition 3.1 [1] *If $\boldsymbol{\alpha}^\perp(\xi) \in C^2(R, R^2)$ and $\boldsymbol{\alpha}^\perp(\xi) = 0$ for $\xi \leq 0$ then*

$$\begin{aligned} \mathbf{A}^\perp(x) &= \boldsymbol{\alpha}^\perp(\xi), & \xi &:= x^0 - z, & n_h &= n_h^{(0)} := 0, & E^z &= E^{z(0)} := 0, \\ \mathbf{u}_h^\perp(x) &= \mathbf{u}_h^{\perp(0)}(\xi) := \frac{-q_h}{m_h c^2} \boldsymbol{\alpha}^\perp(\xi), & u_h^z &= u_h^{z(0)} := \frac{1}{2} \mathbf{u}_h^{\perp(0)2}, & \gamma_h &= \gamma_h^{(0)} := 1 + u_h^{z(0)}, \end{aligned} \quad (21)$$

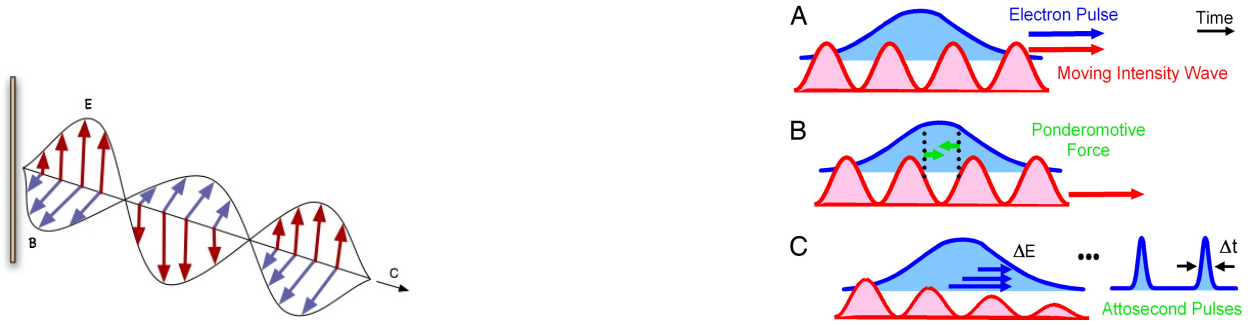


Figure 1: Schematic plots of a linearly polarized transverse wave (left) and of its interaction with a density wave of electrons (right).

which depend on x only through ξ , solve (13-18) and (9).

Let $s_h := \gamma_h - u_h^z$. The difference of eqs. (17-18) gives the equivalent equation

$$\frac{ds_h}{dx^0} = \frac{q_h^2}{2m_h^2 c^4 \gamma_h} (\partial_0 + \partial_z) \mathbf{A}^{\perp 2} - \frac{s_h}{\gamma_h} \frac{q_h E^z}{m_h c^2} \quad (22)$$

The assumptions imply $\frac{ds_h}{dx^0} = 0$, whence $s_h \equiv 1$, which is the main step of the proof. Eqs. (21) give travelling-waves determined solely by the assigned $\boldsymbol{\alpha}^\perp$ and moving in the \hat{z} direction with phase velocity equal to c ; they make up a weak solution if $\boldsymbol{\alpha}^\perp$ is less regular, e.g. $\boldsymbol{\alpha}^\perp(\xi) \in C(\mathbb{R}, \mathbb{R}^2)$ and $\boldsymbol{\alpha}^{\perp'} = \boldsymbol{\epsilon}^\perp$ is continuous except in a finite number of points of finite discontinuities (e.g. at the wavefront). At no time any particle can move in the negative z -direction because $u_h^{z(0)}, \beta_h^{z(0)}$ are nonnegative-definite; the latter are the result of the acceleration by the z -component F_{hm}^z of the magnetic force.

We introduce the following primitives of $\mathbf{u}_h^{(0)}, \gamma_h^{(0)}$:

$$\mathbf{Y}_h(\xi) := \int_0^\xi d\xi' \mathbf{u}_h^{(0)}(\xi'), \quad \Xi_h(\xi) := \int_0^\xi d\xi' \gamma_h^{(0)}(\xi') = \xi + Y_h^3(\xi); \quad (23)$$

As $u_h^{z(0)} \geq 0$, $Y_h^3(\xi)$ is increasing, $\Xi_h(\xi)$ is strictly increasing and invertible.

Proposition 3.2 [1] *Choosing $\mathbf{u}_h \equiv \mathbf{u}_h^{(0)}$, the solution $\mathbf{x}_h^{(0)}(x^0, \mathbf{X})$ of the ODE (7) with the initial condition $\mathbf{x}_h(x^0, \mathbf{X}) = \mathbf{X}$ for $x^0 \leq Z$, and (for fixed x^0) its inverse $\mathbf{X}_h^{(0)}(\mathbf{x}, x^0)$ are*

given by:

$$\begin{aligned}
z_h^{(0)}(x^0, Z) &= x^0 - \Xi_h^{-1}(x^0 - Z), \\
Z_h^{(0)}(x^0, z) &= x^0 - \Xi_h(x^0 - z) = z - Y_h^3(x^0 - z), \\
\mathbf{x}_h^{\perp(0)}(x^0, \mathbf{X}) &= \mathbf{X}^\perp + \mathbf{Y}_h^\perp [x^0 - z_h^{(0)}(x^0, Z)], \\
\mathbf{X}_h^{\perp(0)}(x^0, \mathbf{x}) &= \mathbf{x}^\perp - \mathbf{Y}_h^\perp(x^0 - z).
\end{aligned} \tag{24}$$

These functions fulfill

$$\partial_0 Z_h^{(0)} = -u_h^{z(0)}, \quad \partial_z Z_h^{(0)} = \gamma_h^{(0)}, \quad \partial_Z z_h^{(0)} = \frac{1}{\widetilde{\gamma}_h^{(0)}}, \quad \partial_Z \mathbf{x}_h^{\perp(0)} = -\widetilde{\boldsymbol{\beta}}_h^{\perp(0)}. \tag{25}$$

From (24) it follows that the longitudinal displacement of the h -th type of particles w.r.t. their initial position \mathbf{X} at time x^0 is

$$\Delta z_h^{(0)}(x^0, Z) := z_h^{(0)}(x^0, Z) - Z = Y_h^3[\Xi_h^{-1}(x^0 - Z)]. \tag{26}$$

By (21) the evolution of \mathbf{A}^\perp amounts to a translation of the graph of $\boldsymbol{\alpha}^\perp$. Its value $\check{\boldsymbol{\alpha}}_\perp := \boldsymbol{\alpha}_\perp(\check{\xi})$ at some point $\check{\xi}$ reaches the particles initially located in Z at the time $\check{x}_h^0(\check{\xi}, Z)$ such that

$$\check{x}_h^0 - \check{\xi} = z_h^{(0)}(\check{x}_h^0, Z) \stackrel{(24)_1}{=} \check{x}_h^0 - \Xi_h^{-1}[\check{x}_h^0 - Z] \quad \Leftrightarrow \quad \check{x}_h^0(\check{\xi}, Z) = \Xi_h(\check{\xi}) + Z, \tag{27}$$

in the position $z_h^{(0)}(\check{x}_h^0, Z) = \Xi_h(\check{\xi}) + Z - \check{\xi} = Y_h^3(\check{\xi}) + Z$. The corresponding displacement of these particles is independent of Z and equal to

$$\zeta_h = \Delta z_h^{(0)}[\check{x}_h^0(\check{\xi}, Z), Z] = Y_h^3(\check{\xi}) \tag{28}$$

4 Motion of test particles with arbitrary initial conditions

Eq. (24) describes also the motion of a *single* test particle of charge q_h and mass m_h starting from position \mathbf{X} with velocity $\mathbf{0}$ at sufficiently early time, i.e. before the EM wave arrives. These results for a single test particle can be obtained also by solving the Hamilton-Jacobi equation [8]. Let now $\mathbf{E}^\perp(x) = \boldsymbol{\epsilon}^\perp(x^0 - \mathbf{x} \cdot \mathbf{e})$, $\mathbf{B}^\perp = \mathbf{e} \wedge \mathbf{E}^\perp$ (\mathbf{e} is the unit vector of the direction of propagation of the wave) be an arbitrary free transverse plane EM travelling-wave (we *no longer* require $\mathbf{E}^\perp, \mathbf{B}^\perp$ to vanish for $x^0 - \mathbf{x} \cdot \mathbf{e} < 0$). The *general solution* $\mathbf{x}_h(x^0)$

of the Cauchy problem (6-7) with initial conditions $\mathbf{x}_h(0) = \mathbf{x}_0$, $\frac{d\mathbf{x}_h}{dx^0}(0) = \boldsymbol{\beta}_0$ under the action of such an EM travelling-wave can be now determined by reduction to the previous one as follows. One can do a Poincaré transformation $P = TRB$ to a new reference frame $\underline{\mathcal{F}}$ where the initial velocity and position are zero and the wave propagates in the positive z direction: one first finds the boost B from the initial reference frame \mathcal{F} to a new one \mathcal{F}' where $\boldsymbol{\beta}'_0 = \mathbf{0}$ (this maps the transverse plane electromagnetic wave into a new one), then a rotation R to a reference frame \mathcal{F}'' where the plane wave propagates in the positive z -direction, finally the translation T to the reference frame $\underline{\mathcal{F}}$ where also $\underline{\mathbf{x}}_0 = \mathbf{0}$. Naming \underline{x}^μ the spacetime coordinates and $\underline{A}^\mu, \underline{F}^{\mu\nu}, \dots$ the fields w.r.t. $\underline{\mathcal{F}}$, it is $\frac{d\underline{\mathbf{x}}_h}{d\underline{x}^0}(0) = \mathbf{0}$, $\underline{\mathbf{x}}_h(0) = \mathbf{0}$, and $\underline{\mathbf{E}}^\perp(\underline{x}) = \underline{\boldsymbol{\epsilon}}^\perp(\underline{x}^0 - \underline{z})$, $\underline{\mathbf{B}}^\perp = \hat{\underline{z}} \wedge \underline{\mathbf{E}}^\perp$. Since the part of the EM which is already at the right of the particle at $\underline{x}^0 = 0$ will not come in contact with the particle nor affect its motion, the solution $\underline{\mathbf{x}}_h(\underline{x}^0)$ of the Cauchy problem w.r.t. $\underline{\mathcal{F}}$ does not change if we replace $\underline{\boldsymbol{\epsilon}}^\perp(\underline{x}^0 - \underline{z})$ by the ‘cut’ counterpart $\boldsymbol{\epsilon}_\theta^\perp(\underline{x}^0 - \underline{z}) := \underline{\boldsymbol{\epsilon}}^\perp(\underline{x}^0 - \underline{z}) \theta(\underline{x}^0 - \underline{z})$ (θ stands for the Heaviside step function), see fig. 2. Clearly $\boldsymbol{\epsilon}_\theta^\perp(\xi)$ and $\boldsymbol{\alpha}_\theta^\perp(\xi) := \int_0^\xi d\xi' \boldsymbol{\epsilon}_\theta^\perp(\xi')$ fulfill $\boldsymbol{\epsilon}_\theta^\perp(\xi) = \boldsymbol{\alpha}_\theta^\perp(\xi) = \mathbf{0}$ if $\xi \leq 0$; therefore, denoting as $\mathbf{u}_{h\theta}^{(0)}(\xi), \mathbf{x}_{h\theta}^{(0)}(\underline{x}^0, \mathbf{X}), \dots$ the functions of the previous section obtained choosing $\boldsymbol{\alpha}^\perp(\xi) \equiv \boldsymbol{\alpha}_\theta^\perp(\xi)$, we find $\underline{\mathbf{x}}_h(\underline{x}^0) = \mathbf{x}_{h\theta}^{(0)}(\underline{x}^0, \mathbf{0})$. The solution in \mathcal{F} is finally obtained applying the inverse Poincaré transformation P^{-1} to $\underline{\mathbf{x}}_h(\underline{x}^0)$.



Figure 2: The original $\underline{\mathbf{E}}^\perp$ (left) and its ‘cut’ counterpart (right) as functions of \underline{z} at $\underline{x}^0 = 0$.

5 Integral equations for plane waves

We reformulate the PDE’s as integral equations. Using (11-12) one proves

Proposition 5.1 [1] *For any $\bar{Z} \in \underline{\mathbb{R}}$ eq. (14-15) and (9) are solved by*

$$E^z(x^0, z) = 4\pi \sum_{h=1}^k q_h \tilde{N}_h[Z_h(x^0, z)], \quad \tilde{N}_h(Z) := \int_{\bar{Z}}^Z dZ' \tilde{n}_{h0}(Z'); \quad (29)$$

the neutrality condition $(9)_3$ implies $\sum_{h=1}^k q_h \tilde{N}_h(Z) \equiv 0$.

Formula (29) gives the solution of (14-15) explicitly in terms of the initial densities, up to determination of the functions $Z_h(x^0, z)$.

Using the Green function $G(x^0, z) = \frac{1}{2}\theta(\xi)\theta(\xi_-) = \frac{1}{2}\theta(x^0 - |z|)$ of the d'Alembertian $\partial_0^2 - \partial_z^2 = 4\partial_+\partial_-$ [θ is the Heaviside step function, $\xi_- := x^0 + z$, $\partial_- := \partial/\partial\xi_- = \frac{1}{2}(\partial_0 + \partial_z)$], we can rewrite eq. (16) with initial conditions at $x^0 = X^0$ as the integral equation

$$\mathbf{A}^\perp(x^0, z) - \mathbf{A}_f^\perp(x^0, z) = - \int_{D_x^{X^0}} d^2x' 2\pi \sum_{h=1}^k q_h [n_h \boldsymbol{\beta}_h^\perp](x^0, z') \quad (30)$$

where $\mathbf{A}_f^\perp(x^0, z) = \boldsymbol{\alpha}^\perp(x^0 - z) + \boldsymbol{\alpha}_-^\perp(x^0 + z)$ is determined by the initial conditions,

$$D_x^{X^0} := \{x' \mid X^0 \leq x^0' \leq x^0, |z - z'| \leq x^0 - x^0'\} = \{x' \mid 2X^0 \leq \xi' + \xi', \xi' \leq \xi, \xi'_- \leq \xi_-\}.$$

If beside (9) **we assume** $n_h(0, z) = 0$ **for** $z < 0$, for $x^0 \leq 0$ the EM wave is free and \mathbf{A}^\perp is of the form $\mathbf{A}^\perp(x^0, z) \equiv \boldsymbol{\alpha}^\perp(x^0 - z)$, with $\boldsymbol{\alpha}^\perp(\xi) = 0$ for $\xi \leq 0$. Then in (30) we may choose $X^0 = 0$ [hence $\tilde{n}_{h0}(Z) \equiv n_h(0, Z)$] and set $\mathbf{A}_f^\perp(x^0, z) = \boldsymbol{\alpha}^\perp(x^0 - z)$.

In the Lagrangian description (22) reads $\tilde{\gamma}_h \partial_0 \tilde{s}_h = \tilde{s}_h \tilde{\varepsilon}_h^z + \widetilde{\partial_- \mathbf{u}_h^{\perp 2}}$; the Cauchy problem with initial condition $\tilde{s}_{h0} \equiv 1$ is equivalent to the integral equation

$$\tilde{s}_h = e^{\int_0^{x^0} d\eta \tilde{\mu}_h(\eta, Z)} + \int_0^{x^0} d\eta e^{\int_\eta^{x^0} d\eta' \tilde{\mu}_h(\eta', Z)} \left[\widetilde{\frac{\partial_- \mathbf{u}_h^{\perp 2}}{\tilde{\gamma}_h}} \right](\eta, Z), \quad \tilde{\mu}_h := \frac{-q_h \tilde{E}^z}{m_h c^2 \tilde{\gamma}_h}. \quad (31)$$

$u_h^z, \gamma_h, \boldsymbol{\beta}_h^\perp, \beta_h^z$ can be recovered from s_h, \mathbf{u}_h^\perp through the formulae

$$\begin{aligned} \gamma_h &= \frac{1 + \mathbf{u}_h^{\perp 2} + s_h^2}{2s_h}, & \boldsymbol{\beta}_h^\perp &= \frac{\mathbf{u}_h^\perp}{\gamma_h} = \frac{2s_h \mathbf{u}_h^\perp}{1 + \mathbf{u}_h^{\perp 2} + s_h^2}, \\ u_h^z &= \frac{1 + \mathbf{u}_h^{\perp 2} - s_h^2}{2s_h}, & \beta_h^z &= \frac{u_h^z}{\gamma_h} = \frac{1 + \mathbf{u}_h^{\perp 2} - s_h^2}{1 + \mathbf{u}_h^{\perp 2} + s_h^2}. \end{aligned} \quad (32)$$

The Cauchy problem (7) with initial condition $\mathbf{x}_h(0, \mathbf{X}) = \mathbf{X}$ is equivalent to the integral equations

$$\begin{aligned} \Delta z_e(x^0, Z) &:= z_h(x^0, Z) - Z = \int_0^{x^0} d\eta \beta_h^z[\eta, z_h(\eta, Z)], \\ \mathbf{x}_h^\perp(x^0, \mathbf{X}) - \mathbf{X}^\perp &= \int_0^{x^0} d\eta \boldsymbol{\beta}_h^\perp[\eta, z_h(\eta, Z)] \end{aligned} \quad (33)$$

Summarizing, making use of (13), (11), (29), (32) the evolution of the system is determined by solving the system of integral equations (30), (31), (33)₁ in the unknowns $\mathbf{A}^\perp, s_h, z_h$ [note that, once this is solved, (33)₂ becomes known]. It is natural to try an iterative resolution of the system (30)+(31)+(33)₁ within the general approach of the fixed point theorem: replacing the approximation after k steps [which we will distinguish by the superscript (k)] at the right-hand side (rhs) of these equations we will obtain at the left-hand side (lhs) the approximation after $k+1$ steps [9]. If we are interested in solving the system for a short time interval after the beginning of the interaction between the EM waves and the plasma, and/or the initial densities are not very high, a convenient starting (0-th) step is the zero-density solution $(\mathbf{u}_e^\perp, \tilde{s}_e, z_e) = (\mathbf{u}_e^{\perp(0)}, 1, z_e^{(0)})$. In next section we sketch the next approximation under some simplifying assumptions.

6 Short pulse against a step-density plasma: the sling-shot effect

Henceforth we stick to such small x^0 (small times after the beginning of the interaction) that the motion of ions can be neglected (ions respond much more slowly than electrons because of their much larger mass). We formalize this by considering ions as infinitely massive, so that they remain at rest [$Z_h(x^0, z) \equiv z$ for $h \neq e$], have constant densities, and their contribution to rhs(16) disappears; only electrons contribute: rhs(16) = $2\pi en_e \beta_e^\perp$. Moreover we assume that $\widetilde{n}_{h0}(Z)$ are not only zero for $Z < 0$ but also constant for $Z > 0$: $\widetilde{n}_{e0}(Z) = n_0 \theta(Z)$, etc. (as depicted in fig. 6-left), where n_0 is the initial electron and proton density. Choosing $\bar{Z} = 0$ in (29) we find

$$E^z(x^0, z) = 4\pi \sum_{h=1}^k q_h \widetilde{N}_h[Z_h(x^0, z)] = 4\pi en_0 \{ z \theta(z) - Z_e(x^0, z) \theta[Z_e(x^0, z)] \}. \quad (34)$$

If $z, Z > 0$ this reduces to the known result (see e.g. [10, 11]) that at x^0 the electric force acting on the electrons initially located in Z is of the harmonic type $\widetilde{F}_e^z(x^0, Z) = -4\pi n_0 e^2 \Delta z_e(x^0, Z)$, i.e. proportional to their displacement $\Delta z_e(x^0, Z) := z_e(x^0, Z) - Z$ w.r.t. their initial position.

The corresponding first corrected approximation reads [1]:

$$\begin{aligned} \mathbf{u}_e^{\perp(1)}(x^0, z) - \mathbf{u}_e^{\perp(0)}(x^0 - z) &= -\frac{2\pi e^2}{m_e c^2} \int_{D_x^0} d^2 x' [n_e^{(0)} \beta_e^{\perp(0)}](x'), \\ \tilde{s}_e^{(1)} &= e \tilde{r}_e^{(0)}, \end{aligned} \quad (35)$$

$$\Delta z_e^{(1)}(x^0, Z) = \int_0^{x^0} d\eta \tilde{\beta}_e^{z(1)}(\eta, Z).$$

Here the change from the Eulerian to the Lagrangian description (represented by the tilde) is performed approximating \mathbf{x}_e by $\mathbf{x}_e^{(0)}$; in the second line the integral corresponding to the second term of (31) does not appear because $\partial \mathbf{u}_e^{\perp(0)2} = 0$, and we have abbreviated

$$\tilde{r}_e^{(0)}(x^0, Z) := 4K \int_Z^{x^0} d\eta \frac{z_e^{(0)} \theta[z_e^{(0)}] - Z \theta(Z)}{\tilde{\gamma}_e^{(0)}}(\eta, Z) = 4K V_e^3 [\Xi_e^{-1}(x^0 - Z)],$$

where $K := \frac{\pi e^2 n_0}{m_e c^2}$, $V_e^3(\xi) := \int_0^\xi dy Y_e^3(y)$ [Y_e^3 defined in (23)].

If the EM wave is (19) with $\lambda |\epsilon'_s / \epsilon_s| \leq \delta \ll 1$, setting $w := \frac{e}{k m_e c^2} \epsilon_s$ one finds

$$\boldsymbol{\alpha}_e^\perp \simeq \frac{1}{k} \epsilon_s \boldsymbol{\epsilon}_p^\perp, \quad \mathbf{u}_e^{\perp(0)} \simeq w \boldsymbol{\epsilon}_p^\perp, \quad \mathbf{Y}_e^\perp \simeq -\frac{1}{k} w \boldsymbol{\epsilon}_o^\perp, \quad u_e^{z(0)} \simeq \frac{1}{2} w^2 \boldsymbol{\epsilon}_p^{\perp 2},$$

where $a \simeq b$ means $a = b + O(\delta)$. From (35) one can show [1] that

$$\mathbf{A}_e^{\perp(1)} \simeq \boldsymbol{\alpha}_e^\perp, \quad \mathbf{u}_e^{\perp(1)} \simeq \mathbf{u}_e^{\perp(0)} \quad \text{if} \quad 0 \leq x^0 - z \leq \xi_0, \quad 0 \leq x^0 + z \ll \frac{2\pi}{K\lambda}. \quad (36)$$

(ξ_0 stands for the first maximum point of w, ϵ_s) by showing that the relative difference between the lhs and the rhs is much smaller than 1 in the spacetime region (36)₂. There we find in particular, by (26), (32) and some computation,

$$\beta_e^{z(1)} = \frac{1 + \mathbf{u}_e^{\perp(1)2} - s_e^{(1)2}}{1 + \mathbf{u}_e^{\perp(1)2} + s_e^{(1)2}} \simeq \frac{1 + \mathbf{u}_e^{\perp(0)2} - e^{2r_e^{(0)}}}{1 + \mathbf{u}_e^{\perp(0)2} + e^{2r_e^{(0)}}}, \quad (37)$$

$$\Delta z_e^{(1)}(x^0, Z) \simeq \int_0^{\Xi_e^{-1}(x^0 - Z)} dy [\gamma_e^{(0)} \beta_e^{z(1)}](y), \quad (38)$$

$$0 \leq [\Delta z_e^{(0)} - \Delta z_e^{(1)}](x^0, Z) \simeq G[\Xi_e^{-1}(x^0 - Z)],$$

$$G(\xi) := \int_0^\xi dy g(y), \quad g := \frac{(1 + 2u_e^{z(0)})(e^{2r_e^{(0)}} - 1)}{1 + 2u_e^{z(0)} + e^{2r_e^{(0)}}}, \quad (39)$$

$$0 \leq \frac{\Delta z_e^{(0)} - \Delta z_e^{(1)}}{\Delta z_e^{(0)}}(x^0, Z) \simeq T[\Xi_e^{-1}(x^0 - Z)], \quad T := \frac{G}{Y_e^3}. \quad (40)$$

The last expression is the relative difference between the displacement Δz_e in the zero-density and in the first corrected approximation. Hence the approximation $z_e(x^0, Z) \simeq z_e^{(1)}(x^0, Z) \simeq z_e^{(0)}(x^0, Z)$ may be good only as long as $T[\Xi_e^{-1}(x^0 - Z)] \ll 1$. By (27), the maximum $\alpha^\perp(\xi_0)$ reaches the electrons initially located in Z at the time $\tilde{x}^0(Z) = \Xi_e(\xi_0) + Z$; therefore the approximation $z_e(x^0, Z) \simeq z_e^{(0)}(x^0, Z)$ may be good for all $x^0 \leq \tilde{x}^0(Z)$ only if

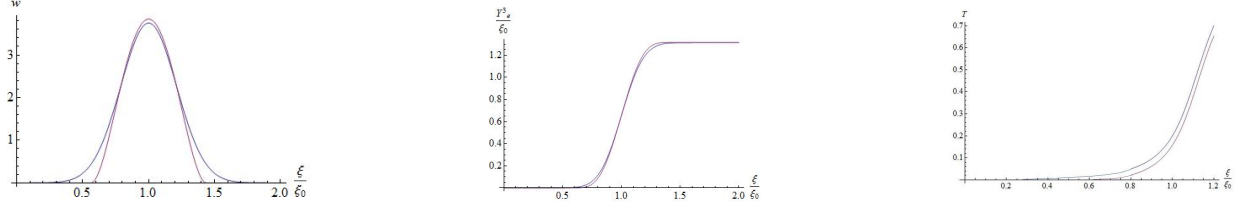
$$T(\xi) \ll 1 \quad 0 \leq \xi \leq \xi_0, \quad 2Y_e^3(\xi_0) + \xi_0 + 2Z \ll \frac{2\pi}{K\lambda}. \quad (41)$$

In particular, (28) with $\tilde{\xi} = \xi_0$ will give a good estimate ζ_e of the displacement of the plasma-surface electrons (those with Z close to zero) if (41) is satisfied.

In [2] ζ_e is used to predict and estimate the *slingshot effect*, i.e. the expulsion of very energetic electrons in the negative z -direction shortly after the impact of a suitable ultra-short and ultra-intense laser pulse in the form of a *pancake* (i.e. a cylinder of radius R and height $l \ll R$) normally onto a plasma. The mechanism is very simple: the plasma electrons in a thin layer - just beyond the surface of the plasma - first are given sufficient electric potential energy by the displacement ζ_e w.r.t. the ions, then after the pulse are pulled back by the longitudinal electric force exerted by the latter and may leave the plasma. Sufficient conditions for this to happen are: 1. $l \ll R$, so that plane wave solutions are sufficiently accurate within the plasma, especially in the forward boost phase; 2. $R \gtrsim 2\zeta$, to avoid trapping of the boosted electrons or even the onset of the *bubble regime* [7]; 3. the EM field inside the pancake is sufficiently intense, and/or n_0 is sufficiently low, so that the longitudinal electric force induces the back-acceleration of the electrons mainly *after* the pulse maximum has overcome them (in phase with the negative ponderomotive force exerted by the pulse in its decreasing stage). Actually we impose the stronger condition that n_0 is sufficiently small in order that (41) be fulfilled and the estimate ζ_e be reliable. As a result, an estimate of the final energy of the electrons initially located at $Z=0$ after the expulsion is [2]

$$H = mc^2 \gamma_{eM}, \quad \gamma_{eM} \simeq 1 + 2K\zeta_e^2. \quad (42)$$

The above conditions are already at hand in several laboratories. The resulting H would be of few MeV implementing those available at the FLAME facility (LNF, Frascati), or at the ILIL laboratory (CNR, Pisa): the pulse energy is a few joules, $\lambda \sim 10^{-4} \text{ cm}$, $\xi_0 \sim 10^{-3} \text{ cm}$, $K\xi_0^2 \sim 1$ (whence $n_0 \sim 10^{18} \text{ cm}^{-3}$); (41) are fulfilled, see the typical plots reported below [the blue, purple curves resp. correspond to a gaussian and to a cut-off polynomial amplitude $w(\xi)$].



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